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Characterizations of Strong Unicity in Approximation Theory

M. W. BARTELT* AND H. W. MCLAUGHLIN

Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York 12181

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1. INTRODUCTION

In this note M denotes a subspace of the complex normed linear space W. An element π in M is called a best approximate to an element f in W if

$$\|f - m\| \geqslant \|f - \pi\|$$

for all m in M; π is a unique best approximate to f if the inequality is strict for all m in M, $m \neq \pi$; π is a strongly unique best approximate to f if there exists a real number $r > 0$ such that

$$\|f - m\| \geqslant \|f - \pi\| + r\|\pi - m\|$$

for all m in M. When M is a Haar subspace of $C(X)$, the space of continuous real valued functions on a compact Hausdorff space X with the supremum norm, Newman and Shapiro [4] have shown that to every f in $C(X)$ there exists a strongly unique best approximate from M. One concludes from Haar's theorem [2] that when M is a finite dimensional subspace of $C(X)$, but not a Haar subspace, there exists at least one f in $C(X)$ to which a best approximate from M is not unique and, hence, not strongly unique.

In the theorems below we characterize those elements of W for which the best approximate from M is strongly unique. This is done by extending a notion introduced by Haar [2]. When M is a finite dimensional subspace of $C(X)$ and X a compact subset of n-dimensional Euclidean space, Haar characterized the best approximate to an element f in $C(X)$ of norm one by means of particular supporting hyperplane to the unit ball in $\langle M, f\rangle$ (the

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linear span of M and f in W). We characterize a strongly unique best approximate in an arbitrary normed linear space W by means of a particular supporting cone to the unit ball in $\langle M, f\rangle$.

In addition, we give two other characterizations of strong unicity, one of which is a "refinement" of the Kolmogorov condition [3] when M is a finite dimensional subspace of $C(X)$.

2. Discussion and Notation

If f belongs to M then $\pi = f$ is the unique best approximate to f from M and it is trivially strongly unique. Henceforth, we assume that f is an arbitrarily chosen but fixed element of W and that $f \notin M$.

We denote by $\langle M, f\rangle$ the linear span of M and f, and by $\langle M, f\rangle^*$ the dual of $\langle M, f\rangle$. Further, for each $\pi \in M$ we let

$$\mathscr{L}_\pi = \{L \in \langle M, f\rangle^* : L(f - \pi) = \| f - \pi \| \text{ and } \| L \| = 1\}.$$

Fixing $\pi \in M$ and letting $B = \{z \in \langle M, f\rangle : \| z \| = \| f - \pi \|\}$ we remark that $\mathscr{L}_\pi$ is exactly the set of continuous linear functionals L defined on $\langle M, f\rangle$ such that $\{z \in \langle M, f\rangle : \operatorname{Re} Lz = \| f - \pi \|\}$ is a supporting plane to B at $f - \pi$. For if $L \in \langle M, f\rangle^*$ is such that $\{z \in \langle M, f\rangle : \operatorname{Re} Lz = \| f - \pi \|\}$ is a supporting plane to B at $f - \pi$ then $\operatorname{Re} Lz \leqslant \| z \|$ for all $z \in \langle M, f\rangle$. Hence, for every complex number a and all $m \in M$ one has

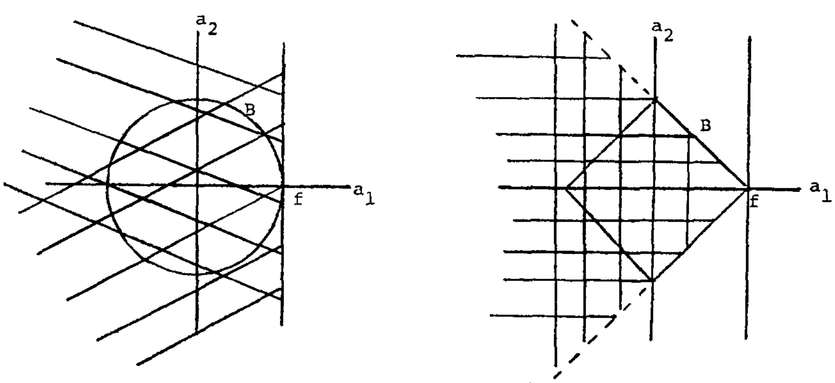
$$\begin{aligned} | L(m + af)|^2 &= \overline{L(m + af)}\, L(m + af) \\ &= L[(\overline{L(m + af)})(m + af)] \\ &= \operatorname{Re} L[\overline{L(m + af)}(m + af)] \\ &\leqslant | L(m + af)| \, \| m + af \|. \end{aligned}$$

Thus, $\| L \| \leqslant 1$. Thus, since $\operatorname{Re} L(f - \pi) = \| f - \pi \|$, one actually has $L(f - \pi) = \| f - \pi \|$. Thus, $L \in \mathscr{L}_\pi$. The converse follows immediately.

For each $\pi \in M$ we write $K_\pi = \{z \in \langle M, f\rangle : \operatorname{Re} Lz \leqslant \| f - \pi \| \; \forall L \in \mathscr{L}_\pi\}$. The set K_π is the supporting cone to the ball in $\langle M, f\rangle$ of radius $\| f - \pi \|$ at the point $f - \pi$. Further, for each $\pi \in M$ we denote by L_π that element of $\langle M, f\rangle^*$ defined by $L_\pi(m + af) = a \| f - \pi \|$ for all $m \in M$ and for all complex numbers a. It follows that $\| L_\pi \| \geqslant 1$.

Haar's result [2] (as is well known) can be stated in the setting of an arbitrary normed linear space as follows: *The element $\pi \in M$ is a best approximate to $f \in W$ if and only if $\| L_\pi \| = 1$, i.e., if and only if $L_\pi \in \mathscr{L}_\pi$; further* *if π is a best approximate to f then it is unique if and only if the hyperplane $\{z \in \langle M, f \rangle : \operatorname{Re} L_\pi z = \| f - \pi \|\}$ intersects the ball in $\langle M, f \rangle$ of radius $\| f - \pi \|$ at precisely $f - \pi$.*

To illustrate Haar's ideas and give a geometric interpretation of the theorems below we discuss two examples. In both Examples 1 and 2, W is taken to be $\{(a_1, a_2): a_1, a_2 \text{ are real}\}$ (with the usual rules for addition and scalar multiplication), $M = \{(0, a_2): a_2 \text{ is real}\}$, and $f = (1, 0)$. In Example 1 the norm is the l_2 or Euclidean norm, i.e., $\|(a_1, a_2)\| = (a_1^2 + a_2^2)^{1/2}$ and in Example 2 the norm is taken to be the l_1 norm, i.e., $\|(a_1, a_2)\| = | a_1 | + | a_2 |$. The vertical line through f in each case represents the hyperplane defined by L_0, and the closed curves B denote the unit circle in $\langle M, f \rangle$. In both cases 0 is a unique best approximate to f. In Example 1 zero is not a strongly unique best approximate to f; in Example 2 it is. In each case the shaded areas represent the supporting cone K_0. Roughly speaking the theorems below indicate that if the unit ball B is "tangent" to the hyperplane defined by L_0 then 0 is not a strongly unique best approximate, otherwise it is. Or as Theorems 3 and 4 assert, in the case when M is a finite dimensional subspace of a real normed linear space, 0 is a strongly unique best approximate to f if and only if the supporting cone K_0 intersects the hyperplane defined by L_0 at exactly one point, namely f; see the foregoing examples.



Example 1. Example 2.

FIGURE 1

In the proof of Theorem 1 below we assume that f is normalized so that $1 = \|f\| = \inf_{m \in M} \|f - m\|$. The following proposition shows that this normalization assumption can be made without loss of generality.

PROPOSITION 1. *If the best approximate to f from M is strongly unique then so is the best approximate to every element of $\langle M, f \rangle$. More precisely,* *suppose that for some* $\pi \in M$ *there exists* $r > 0$ *such that* $\|f - m\| \geq \|f - \pi\| + r\|\pi - m\| \; \forall m \in M$. *Then, letting* a *denote a complex number and* $m_0 \in M$ *and defining* $\pi_1 = a\pi + m_0$ *one obtains the inequality* $\|(af + m_0) - m\| \geq \|(af + m_0) - \pi_1\| + r\|\pi_1 - m\| \; \forall m \in M$.

Proof. The case $M = \{0\}$ is trivial. Assume that $M \neq \{0\}$. Since for all $m \in M$, $\|f - \pi\| + r\|\pi - m\| \leq \|f - m\| \leq \|f - \pi\| + \|\pi - m\|$, one has $0 < r \leq 1$. Hence, if $a = 0$ the result follows trivially and if $a \neq 0$, then for all $m \in M$,

$$\begin{aligned}\|(af + m_0) - m\| &= |a| \, \|f - 1/a(m - m_0)\| \\ &\geq |a| \, \|f - \pi\| + |a| \, r \|\pi - 1/a(m - m_0)\| \\ &= \|(af + m_0) - \pi_1\| + r\|\pi_1 - m\|,\end{aligned}$$

where we have used the strong unicity of π in the estimate.

3. Strong Unicity in Arbitrary Normed Linear Spaces

For convenience we reiterate our assumptions: M denotes an arbitrary subspace of the complex normed linear space W, $f \in W$, and $f \notin M$. Theorem 1 below is due to Wulbert [6, 7], who proved it in the real case. A proof is given for completeness.

Theorem 1. *There exists an element* π *in* M *and a real number* $r > 0$ *such that*

$$\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m) \geq r\|m\| \; \forall m \in M$$

if and only if

$$\|f - m\| \geq \|f - \pi\| + r\|\pi - m\| \; \forall m \in M.$$

Proof. By Proposition 1 we assume without loss that $\pi = 0$ and $\|f\| = 1$. Suppose first that the real number r satisfies

$$\sup_{L \in \mathscr{L}_0} \operatorname{Re} L(m) \geq r\|m\| \; \forall m \in M.$$

Then for every $m \in M$ one has

$$\begin{aligned}\|f - m\| &= \sup_{\substack{L \in \langle M, f\rangle^* \\ \|L\| = 1}} |L(f - m)| \geq \sup_{L \in \mathscr{L}_0} |L(f - m)| \geq \sup_{L \in \mathscr{L}_0} \operatorname{Re} L(f - m) \\ &= 1 + \sup_{L \in \mathscr{L}_0} \operatorname{Re} L(-m) \geq 1 + r\|m\|.\end{aligned}$$

Conversely, suppose that $\|f - m\| \geqslant 1 + r\|m\| \ \forall m \in M$. Let m be an arbitrarily chosen but fixed element of M, and define $L' \in \langle m, f\rangle^*$ by $L'(am + bf) = ar\|m\| + b$ for all complex numbers a and b. If $b = 0$ then since $r \leqslant 1$, one has $|L'(am)| = |a|\, r\|m\| \leqslant \|am\|$; and if $b \neq 0$ then

$$\begin{aligned} |L'(am+bf)| &= |ar\|m\| + b| = |b|\,(r\|-(a/b)m\| + 1) \\ &\leqslant |b|\,\|f + (a/b)m\| = \|am + bf\|, \end{aligned}$$

where we have used the estimate $\|f - m\| \geqslant 1 + r\|m\| \ \forall m \in M$. Thus, $\|L'\| \leqslant 1$. But $L'f = 1$, so $\|L'\| = 1$. Hence, by the Hahn–Banach theorem we may assume without loss that $L' \in \langle M, f\rangle^*$ and that $L' \in \mathscr{L}_0$. Since $\operatorname{Re} L'(m) = r\|m\|$ one has $\sup_{L \in \mathscr{L}_0} \operatorname{Re} L(m) \geqslant r\|m\|$.

Theorem 2 below gives another characterization of strong unicity. Its proof follows from Theorem 1 and the following lemma.

LEMMA 2. *The set $K_\pi \cap M$ is bounded if and only if there exists $r > 0$ such that* $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m) \geqslant r\|m\| \ \forall m \in M$.

Proof. Suppose first that $K_\pi \cap M$ is bounded and let $R > 0$ be chosen such that for all $m \in M$, $\|m\| = 1$, one has $Rm \notin K_\pi \cap M$. Thus, $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(Rm) > \|f - \pi\|$ and, hence, letting $r = \|f - \pi\|/R$ one has $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m) \geqslant r\|m\| \ \forall m \in M$. Conversely, suppose that there exists $r > 0$ such that $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m) \geqslant r\|m\| \ \forall m \in M$. Then if $m \in M$ and $\|m\| > \|f - \pi\|/r$ one has $\sup \operatorname{Re} L(m) > \|f - \pi\|$, which means that $m \notin K_\pi$ and, hence, $m \notin K_\pi \cap M$. Thus, $K_\pi \cap M$ is contained in a sphere, centered at the origin, of radius $\|f - \pi\|/r$.

THEOREM 2. *There exists an element $\pi \in M$ and a real number $r > 0$ such that*

$$\|f - m\| \geqslant \|f - \pi\| + r\|\pi - m\| \ \forall m \in M$$

if and only if $K_\pi \cap M$ is bounded.

COROLLARY. *$K_\pi \cap M$ is bounded for at most one element $\pi \in M$.*

In Example 1 the unit ball in l_p for $p > 2$ contains the l_2 unit ball. Clearly then the l_p unit ball will also be "tangent" to the hyperplane defined by L_0 and 0 is not a strongly unique best approximate to f in l_p. In general, let $\|\cdot\|_1$ be a norm on W and assume without loss that $\|f\|_1 = 1$ and that 0 is a best approximate to f from M in the norm $\|\cdot\|_1$. Let $(W, \|\cdot\|)$ denote W with the norm $\|\cdot\|$. If $\|\cdot\|_2$ is another norm on W, then the unit ball in $(W, \|\cdot\|_2)$ is contained in the unit ball in $(W, \|\ \|_1)$ if and only if $\|w\|_1 \leqslant \|w\|_2 \ \forall w \in W$. If $\|f\|_2 = 1$ and $\|w\|_1 \leqslant \|w\|_2 \ \forall w \in W$, then 0 is also a best approximate to f in the norm $\|\cdot\|_2$. As an application of the last theorem we show the same result for strong uniqueness.

COROLLARY. *Let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two norms on* W. *Assume that*

(i) 0 *is a strongly unique best approximate to* f *in* $(W, \|\cdot\|_1)$,

(ii) $\|w\|_1 \leqslant \|w\|_2 \ \forall w \in W$,

(iii) $\|f\|_2 = 1$.

Then 0 *is a strongly unique best approximate to* f *in* $(W, \|\cdot\|_2)$.

Proof. For any linear functional L on $\langle M, f\rangle$ with $\|L\|_1 < \infty$, condition (ii) implies that $\|L\|_2 \leqslant \|L\|_1$. Let $\mathscr{L}_0^i = \{L: Lf = 1 = \|L\|_i\}$ for $i = 1, 2$. If $L \in \mathscr{L}_0^1$, then $Lf = 1 \leqslant \|L\|_2 \leqslant \|L\|_1$, and, hence, $\mathscr{L}_0^1 \subseteq \mathscr{L}_0^2$. Therefore, the corresponding supporting cones K_1 and K_2 satisfy $K_2 \subset K_1$. By the last theorem $K_1 \cap M$ is bounded. Therefore, $K_2 \cap M$ is bounded and the last theorem completes the proof.

Theorems 3 and 4 taken together give another characterization of strong unicity when M is finite dimensional.

THEOREM 3. *If there exists an element* $\pi \in M$ *and a real number* $r > 0$ *such that*

$$\|f - m\| \geqslant \|f - \pi\| + r\|\pi - m\| \ \forall m \in M$$

then the set

$$\{z \in \langle M, f\rangle : \operatorname{Re} L_\pi z = \|f - \pi\|\} \cap K_\pi$$

consists of exactly those elements of the form $x = (1 + ia)(f - \pi)$ *where* a *denotes an arbitrary real number.*

Proof. If $x = (1 + ia)(f - \pi)$ then clearly $x \in \{z \in \langle M, f\rangle : \operatorname{Re} L_\pi z = \|f - \pi\|\} \cap K_\pi$ (independently of the hypothesis).

Now suppose $x \in \{z \in \langle M, f\rangle : \operatorname{Re} L_\pi z = \|f - \pi\|\} \cap K_\pi$. Since $\operatorname{Re} L_\pi x = \|f - \pi\|$ one has $x = m + (1 + ia)f$ for some m and some real number a. Since $x \in K_\pi$ one has $\|f - \pi\| \geqslant \operatorname{Re} Lx = \operatorname{Re} L(m + (1 + ia)f) = \operatorname{Re} L(m + (1 + ia)\pi) + \|f - \pi\|$ for all $L \in \mathscr{L}_\pi$. Thus, $\operatorname{Re} L(m + (1 + ia)\pi) \leqslant 0 \ \forall L \in \mathscr{L}_\pi$. Hence, $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m + (1 + ia)\pi) \leqslant 0$. From Theorem 1 one concludes that $m = -(1 + ia)\pi$.

THEOREM 4. *Let* M *be finite dimensional. If there exists an element* $\pi \in M$ *such that*

$$\{z \in \langle M, f\rangle : \operatorname{Re} L_\pi(z) = \|f - \pi\|\} \cap K_\pi$$

consists of exactly those elements of the form $x = (1 + ia)(f - \pi)$ *where* a *is an arbitrary real number then there exists a real number* $r > 0$ *such that*

$$\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \; \forall m \in M.$$

Proof. If there exists no such r, then from Theorem 1 one concludes that $\inf_{m \in M, \|m\|=1} \sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m) \leqslant 0$. Since the unit ball in M is compact there exists $m_0 \in M$, $\| m_0 \| = 1$ such that $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} L(m_0) \leqslant 0$. Hence, $m_0 + (1 + ia)(f - \pi)$ belongs to $\{z \in \langle M, f \rangle : \operatorname{Re} L_\pi(z) = \| f - \pi \|\} \cap K_\pi$ for all a. This violates the hypothesis.

The following example shows that Theorem 4 is, in general, no longer true if the hypothesis that M be finite dimensional is deleted.

EXAMPLE 3. Let W be the real normed linear space consisting of all sequences of the form $a = \{a_i\}_1^\infty$, where a_i is real $\forall i$ and $\sup_{1 \leqslant i < \infty} | a_i | < \infty$. Addition and scalar multiplication are defined component-wise and $\| a \| = \sup_{1 \leqslant i < \infty} | a_i |$. Let M be the subspace consisting of all sequences such that $a_{3i} = -ia_{3i-1} = ia_{3i-2}$ $(i = 1, 2, ...)$. A general element of M then has the form $(b_1, -b_1, b_1, b_2, -b_2, 2b_2, ...)$. Let $f = \{f_i\}_1^\infty = (1, 1, ...)$, i.e., $f_i = 1 \; \forall i$. Clearly 0 is the unique best approximate to f from M. For an arbitrary $g = \{g_i\} \in \langle M, f \rangle$ the element L_0 $(\in \mathscr{L}_0)$ has the form $L_0 g = \frac{1}{2}(g_1 + g_2)$. It is easily verified that $L_0(M) = 0$, $L_0 f = 1 = \| L_0 \|$, and, hence, $\{z \in \langle M, f \rangle : L_0 z = 1\} = f + M$. For each i $(1 \leqslant i \leqslant n)$ we define $L^{(i)} \in \mathscr{L}_0$ by $L^{(i)}(g) = g_i$, $g = \{g_i\}_1^\infty \in \langle M, f \rangle$. If $g \in \{z \in \langle M, f \rangle : L_0 z = 1\} \cap K_0$ then g has the form $(1 + b_1, 1 - b_1, 1 + b_1, 1 + b_2, 1 - b_2, 1 + 2b_2, ...)$ with $1 + b_i \leqslant 1$ and $1 - b_i \leqslant 1$ $(1 \leqslant i \leqslant n)$, and, hence, $b_i = 0 \; \forall i$. Thus, $\{z \in \langle M, f \rangle : L_0 z = 1\} \cap K_0 = \{f\}$. To show that 0 is not a strongly unique best approximate to f we define an infinite sequence $\beta^n = \{b_i^n\}_{i=1}^\infty$ $(n = 1, ...)$ on the unit ball in M by

$$b_{3i}^n = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases} \qquad (i, n = 1, ...).$$

The sequence $\{\beta^n\}_{n=1}^\infty$ has the form

$$\beta^1 = (1, -1, 1, 0, 0, 0, ...),$$

$$\beta^2 = (0, 0, 0, \tfrac{1}{2}, -\tfrac{1}{2}, 1, 0, 0, ...).$$

Thus, $\| f - \beta^n \| = 1 + 1/n$ $(n = 1, 2, ...)$ which means that there exists no $r > 0$ such that $\| f - \beta^n \| \geqslant 1 + r \| \beta^n \|$.

4. Strong Unicity in $C(X)$

In this section X denotes a nonempty compact Hausdorff space and $C(X)$ denotes the vector space of continuous complex valued function defined on X with the supremum norm. As before M denotes a subspace of $C(X)$ and f denotes an arbitrary element of $C(X)$ which does not belong to M.

The following lemma can be found in [5] and [8].

LEMMA 3. *Let M be a subspace of $C(X)$ of dimension n $(n \geqslant 1)$, and let L be a nonzero linear functional defined on M. Then there exist points $p_1,..., p_r$ in X $(r \leqslant 2n - 1)$ and nonzero constants $\alpha_1,..., \alpha_r$ such that $L = \sum_{i=1}^{r} \alpha_i L_{p_i}$, where $L_{p_i}(g) = g(p_i)\ \forall g \in M$ $(1 \leqslant i \leqslant r)$; and $\sum_{i=1}^{r} | \alpha_i | = \| L \|$. Further, if $g^* \in M$, $\| g^* \| = 1$ and $Lg^* = \| L \|$ then $g^*(p_i) = \overline{\operatorname{sgn} \alpha_i}$ $(1 \leqslant i \leqslant r)$.*

The next theorem which characterizes strongly unique best approximates can be thought of as a generalization of the Kolmogorov condition [3] for when $r = 0$ it reduces to this well known condition.

THEOREM 5. *Let $\pi \in M$ and $A = \{x \in X: | f(x) - \pi(x)| = \| f - \pi \|\}$. Then the real number $r > 0$ satisfies*

$$\max_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] \geqslant r \| f - \pi \| \, \| m \| \ \forall m \in M$$

if and only if

$$\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \ \forall m \in M.$$

Proof. We assume first that

$$\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \ \forall m \in M.$$

Let $L \in \mathscr{L}_\pi$ and $m \in M$. By restricting L to $\langle m, f - \pi \rangle$ one has from Lemma 3 that there exists $x_1,..., x_r \in A$ $(r \leqslant 3)$ such that

$$Lm = \sum_{i=1}^{r} | \alpha_i | \, \| f - \pi \|^{-1} (\overline{f(x_i) - \pi(x_i)})\, m(x_i).$$

Thus,

$$\operatorname{Re} Lm \leqslant \max_{x \in A} \operatorname{Re}[\| f - \pi \|^{-1} (\overline{f(x) - \pi(x)})\, m(x)].$$

One then obtains

$$\max_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] \geqslant \| f - \pi \|$$

$$\cdot \sup_{L \in \mathscr{L}_\pi} \operatorname{Re} Lm \geqslant r \| f - \pi \| \, \| m \|$$

where the last estimate is obtained from Theorem 1.

Conversely if

$$\max_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] \geqslant r \| f - \pi \| \, \| m \|$$

for all $m \in M$ one defines $L_x \in \mathscr{L}_\pi$ $(x \in A)$ by

$$L_x g = \| f - \pi \|^{-1} (\overline{f(x) - \pi(x)})\, g(x) \ \forall g \in \langle M, f \rangle,$$

to obtain $\sup_{L \in \mathscr{L}_\pi} \operatorname{Re} Lm \geqslant \max_{x \in A} \operatorname{Re} L_x m \geqslant r \| m \| \ \forall m \in M$. By Theorem 1 then $\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \ \forall m \in M$.

Remark 1. If the subspace M of the above theorem is assumed to be finite dimensional then

$$\max_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] > 0 \ \forall m \in M,\ m \neq 0,$$

if and only if there exists $r > 0$ such that

$$\max_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] > r \| f - \pi \| \, \| m \| \ \forall m \in M.$$

This is easily verified by noting that the unit ball in $\langle M, f \rangle$ is compact. In other words a necessary and sufficient condition that there exists $r > 0$ such that

$$\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \ \forall m \in M$$

is that

$$\sup_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] > 0 \qquad \text{for all} \quad m \in M, \quad m \neq 0.$$

The following example shows that there exists an infinite dimensional subspace M of some $C(X)$ such that there is an element $f \in C(X)$, $\| f \| = 1$, such that $\max_{x \in A} f(x)\, m(x) > 0 \ \forall m \in M$, $m \neq 0$, but the zero function which is a unique best approximate to f is not a strongly unique best approximate.

EXAMPLE 4. Let $X = [-2, 2]$ and $f(x)$, $x \in [-2, 2]$, be an even function defined on $[0, 2]$ by

$$f(x) = \begin{cases} \frac{1}{2} & 0 \leqslant x \leqslant \frac{1}{2}, \\ x & \frac{1}{2} \leqslant x \leqslant 1, \\ 1 & 1 \leqslant x \leqslant 2. \end{cases}$$

Let M be the odd polynomials, i.e.,

$$M = \left\{ \sum_{k=1}^{n} a_k x^{2k-1} \colon a_k \text{ real } (k = 1,\dots, n)\ n = 1, 2,\dots \right\}.$$

Since f is an even function, 0 is a unique best approximate to f from M and clearly $\max_{x \in A} f(x)\, m(x) > 0$ for all $m \in M$, $m \neq 0$. Now let $g(x)$, $x \in [-2, 2]$ be an odd function defined on $[0, 2]$ by

$$g(x) = \begin{cases} x & 0 \leqslant x \leqslant \frac{1}{2}, \\ -x + 1 & \frac{1}{2} \leqslant x \leqslant 1, \\ 0 & 1 \leqslant x \leqslant 2. \end{cases}$$

Then $\| g \| = \frac{1}{2}$ and $\| g - f \| = 1$. So if $\{m_n\}_{n=1}^{\infty}$ is a sequence in M such that $\lim_{n\to\infty} \| g - m_n \| = 0$ then $\lim_{n\to\infty} \| m_n \| = \frac{1}{2}$ and $\lim_{n\to\infty} \| f - m_n \| = 1$. The existence of such a sequence is ensured by the Weierstrass theorem. Thus, there exists no $r > 0$ such that $\| f - m_n \| \geqslant 1 + r \| m_n \|$, $n = 1, 2,\dots$, i.e., 0 is not a strongly unique best approximate to f.

The next theorem is a generalization of a result of Rivlin and Shapiro [5].

THEOREM 6. *Let* $\pi \in M$ *and* $A = \{x \in X \colon | f(x) - \pi(x)| = \| f - \pi \|\}$. *Let* $g_1, \dots, g_n$ *be a basis for the subspace* M *of* $C(X)$, *and let* con E *denote the convex hull in complex n-space of*

$$E = \{((\overline{f(x) - \pi(x)})\, g_1(x),\dots, (\overline{f(x) - \pi(x)})\, g_n(x)) \colon x \in A\}.$$

Then the origin in complex n-space belongs to the interior of con E *if and only if there exists* $r > 0$ *such that*

$$\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \ \forall m \in M.$$

Proof. That there exists $r > 0$ such that

$$\| f - m \| \geqslant \| f - \pi \| + r \| \pi - m \| \ \forall m \in M$$

is equivalent to

$$\max_{x \in A} \operatorname{Re}[(\overline{f(x) - \pi(x)})\, m(x)] > 0 \ \forall m \in M,\ m \neq 0,$$

which in turn is equivalent to

$$\max_{x\in A} \operatorname{Re} \sum_{i=1}^{n} a_i(\overline{f(x) - \pi(x)})\, g_i(x) > 0$$

for all choices of the complex numbers $a_1, \ldots, a_n$ not all of which are zero. Since A is a compact set so is E. And whenever $(a_1, \ldots, a_n)$ are arbitrarily chosen complex numbers not all zero, the set of all vectors $(z_1, \ldots, z_n)$ in complex n-space satisfying

$$\operatorname{Re} \sum_{i=1}^{n} a_i z_i = \max_{x\in A} \operatorname{Re} \sum_{i=1}^{n} a_i(\overline{f(x) - \pi(x)})\, g_i(x)$$

is a supporting plane of con E. Further every supporting hyperplane is obtained in this manner. Thus,

$$\max_{x\in A} \operatorname{Re}(\overline{f(x) - \pi(x)})\, m(x) > 0 \ \forall m \in M,\ m \neq 0,$$

is equivalent to: the zero vector (obtained by setting $z_1 = \cdots = z_n = 0$), belongs to every half space containing E, and lies on no supporting hyperplane of con E, which, since E is compact, is equivalent to the fact that the zero vector belongs to the interior of con E.

Remark 2. If π is a strongly unique best approximate to $f \in C(X)$, from an n-dimensional subspace M of $C(X)$, then, since the convex hull of E has an interior, E must consist of at least $n + 1$ points. One might guess that when X is a compact interval of the real line, the error function $f - \pi$ equioscillates. The following example shows that in general this is not true.

EXAMPLE 5. Let $X = [-1, 1]$, $g(x) = x$ $(-1 \leqslant x \leqslant 1)$, $M = \langle g \rangle$, and $f(x) \equiv 1$ $(-1 \leqslant x \leqslant 1)$. Then for all $m \in M$ $\| f - m \| = 1 + \| m \|$ so 0 is a strongly unique best approximate to f from M. However, the error function is $f(x) - 0 \equiv 1$ $(-1 \leqslant x \leqslant 1)$ which never takes on a negative value and, hence, cannot equioscillate.

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